Journal of Thermophysics and Heat Transfer
Vol. 10, No. 1, January–March 1996

Exponential Box-Schemes for Boundary-Layer Flows with Blowing

Vladimir V. Riabov*
Worcester Polytechnic Institute, Worcester, Massachusetts 01609-2280
and
Victor P. Provotorov†
Central Aerohydrodynamics Institute, Moscow 140160, Russia

A two-point second-order uniform box-scheme has been developed for numerical analysis of the gas flow in the boundary layer under the conditions of intensive blowing on the body surface. An effective regularization algorithm has been created. The numerical analysis indicates the second order of the uniform convergence of this box-scheme. Flowfield parameters in the boundary layer near the stagnation point of a blunt body were calculated for different magnitudes of blowing and temperature factor. Numerical results have been compared with earlier obtained data by the three-point exponential box-scheme.

Nomenclature

C = constant, Eq. (33)
f = function, Eqs. (1) and (2)
H = total enthalpy
h = grid-cell size
j = parameter of the symmetry
P = order of uniform convergence, Eq. (33)
r_w = distance from the axis of symmetry to generatrix of the body
S = H/H_δ, enthalpy ratio
t_w = temperature factor
u = velocity component along the generatrix of the body
$\bar{u}$ = u/u_δ, normalized value of the velocity component u
x = coordinate along the generatrix of the body
y = coordinate normal to the body surface
α = coefficient, Eq. (6)
β = $(1 + j)^{-1}$, Falkner–Skan constant
γ = coefficient, Eq. (6)
ε = small coefficient, Eq. (5)
η = $u_\delta r_w^j \cdot \int_0^y \rho \, dy/(2\xi)^{1/2}$, normalized coordinate along the normal
μ = viscosity
ξ = $\int_0^x \rho_w \mu_w u_\delta r_w^{2j} \, dx$, normalized coordinate along the generatrix
ρ = density
σ = 0.72, Prandtl number

Subscripts
w = wall conditions
δ = external boundary-layer conditions
ε = parameters estimated at the certain magnitude of the coefficient ε
$+$ = parameters estimated at the midpoint of the $(i + 1)$th grid cell
$-$ = parameters estimated at the midpoint of the ith grid cell

Superscripts
(1) = index of the derivative approximation
(2) = index of the function approximation
$'$ = differentiation along the variable η

Introduction

Many problems of aerodynamics and thermophysics come to solving differential equations with small coefficients at the highest derivative. The latter leads to the formation of regions with small linear dimensions where gradients of the functions are large. Nonuniform convergence or even divergence of numerical solutions takes place in the numerical analysis of such classical problems by traditional box-schemes.[1,2] In this study, gas flow parameters in a boundary layer under the condition of blowing on the body surface are analyzed.

From a mathematical point of view, the increase of the flow rate of blowing gas is equivalent to the existence of a small coefficient at the highest derivative in the boundary-layer equations.[2] As a result, a new sublayer with large gradients of temperature and velocity is created.

The gas flow in the boundary layer at the stagnation point of a blunt body is studied using a two-point uniform exponential box-scheme. The identical problem was considered by El-Mistikawy and Werle[3] using a three-point exponential box-scheme. In order to apply our two-point scheme, a more effective regularization algorithm is developed. The improved matrix variant[4–6] of the regularization is applied.

Gas Blowing into a Boundary Layer

Consider the flow of a perfect gas in the boundary layer near the stagnation point of a blunt body with uniform blowing at the surface. A viscosity coefficient is assumed to be linearly proportional to the gas temperature and the coefficient of proportionality is the Chapman–Rubesin parameter.[7,8] Then the system of boundary-layer equations, considering heat transfer on the body surface, will acquire the following form[2,5]:

$$\bar{u}'' + f\bar{u}' + \beta(S + 1 - \bar{u}^2) = 0 \qquad (1)$$

$$f' = \bar{u}, \qquad S'' + \sigma f S' = 0 \qquad (2)$$

where the constant[8] β characterizes the pressure gradient in inviscid flow; j = 0 or 1 in plane and axisymmetric cases correspondingly.

Received March 10, 1995; revision received Aug. 28, 1995; accepted for publication Sept. 1, 1995; presented as Paper 96-0608 at the AIAA 34th Aerospace Sciences Meeting and Exhibit, Reno, NV, Jan. 15–18, 1996.

*Visiting Associate Professor, Department of Mechanical Engineering, 100 Institute Road. Member AIAA.

†Senior Research Scientist, Rarefied Gas Dynamics Branch, Aerothermodynamics Division, Zhukovsky-3.

Boundary conditions are the following:
On the surface of the body ($\eta = 0$) considering the gas blowing

$$f = f_w = \text{const}, \qquad \bar{u} = 0, \qquad S = S_w \tag{3}$$

On the external boundary of the layer ($\eta \to \infty$)

$$\bar{u} = 1, \qquad S = 0 \tag{4}$$

In Eq. (3) the parameter f_w characterizes the mass flow rate of the blowing gas. The increase of the rate leads to a new type of mathematical problems,[2] i.e., the solution of the equations with small coefficients at the highest derivative. Special box-schemes should be used in order to solve such problems. These are schemes with uniform convergence or exponential schemes.[1,3,5,6]

The exponential box-scheme has been developed by El-Mistikawy and Werle[3] for numerical solution of the Faulkner–Scan Eqs. (1) and (2) at $S = 0$ under the conditions of intensive blowing. The uniform three-point scheme[3] has the second order of convergence when using a variable size of the grid cells. As the authors of the study[3] indicated, their method of calculations contains all of the positive features of both exponential as well as two-point box-schemes. However, their method does not have all of the advantages of the two-point box-schemes, i.e., when their scheme is applied to solve the equation system and some of the equations should or could be solved by the traditional box-scheme.[9] The principal advantages of the two-point box-schemes are 1) using this approach,[10–14] any type of boundary conditions estimated accurately; 2) algorithmization of the grid-cell-size changes is very simple; and 3) fluxes of the flow parameters are calculated without additional procedure and the approximation error of the fluxes is the same as that of other terms of the equations. The two-point exponential box-scheme developed has the second order of uniform convergence. In the case considered previously, it is obviously better if the traditional box-scheme is a two-point one as, e.g., in studies of Keller,[9] Denisenko and Provotorov,[10] Provotorov and Riabov,[5] and Riabov and Provotorov.[11]

In this study, the two-point exponential box-scheme has been developed and analyzed. Its regularization algorithm is described.

Model Equation

Consider the model equation

$$\varepsilon u'' + a u' - b u = d \tag{5}$$

Here the parameter ε can accept very small magnitudes, and $a \geq 0$, $b \geq 0$. Consider two neighboring grid cells $(i, i + 1)$ of the box-scheme.[3,6] The values of the parameters in the cell $(i + 1)$ will be marked by $(+)$, and in the cell (i) by $(-)$. Assume that coefficients in Eq. (5) are constant in half-intervals $(i, i + \frac{1}{2})$ and $(i + \frac{1}{2}, i + 1)$ and they are equal their values correspondingly in the cells (i) and $(i + 1)$. Using the technique of El-Mistikawy and Werle,[3] the solution of Eq. (5) with constant coefficients is the following:

$$u = Ae^{\alpha\eta} + Be^{\gamma\eta} + \psi, \qquad \psi = \frac{d}{\varepsilon\alpha\gamma}$$

$$\alpha = -\frac{a}{2\varepsilon} + \sqrt{\frac{a^2}{4\varepsilon^2} + \frac{b}{\varepsilon}} \tag{6}$$

$$\gamma = -\frac{a}{2\varepsilon} - \sqrt{\frac{a^2}{4\varepsilon^2} + \frac{b}{\varepsilon}}$$

where A and B are arbitrary constants.

The solution (6) is used to obtain the box-scheme characteristics. Consider that the functions as well as the derivatives are continuous in the cell $(i + \frac{1}{2})$. These conditions offer two linear relations for the coefficients A_+, B_+, A_-, and B_-. Besides, using Eq. (6), the values of the functions $\{u^{(2)}\}$ and the derivatives $\{u^{(1)}\}$ will be found in the cells (i) and $(i + 1)$. These conditions determine four linear relations for the coefficients A_+, B_+, A_-, and B_- through the values of the functions $u^{(2)}$ and the derivatives $u^{(1)}$ in the cells (i) and $(i + 1)$. Solving this system of four equations in connection to the coefficients A_+, B_+, A_-, and B_-, and substituting their values in the first two equations connected to two linear relations, the box-scheme will be obtained. The following new parameters are introduced: the parameter h is a grid-cell size of the interval $(i, i + 1)$; $S_\pm = [\exp(\pm\gamma_\pm h)(\alpha_\pm - \gamma_\pm)]^{-1}$; $r_\pm = [\exp(\pm\alpha_\pm h)(\gamma_\pm - \alpha_\pm)]^{-1}$.

Finally, at $1 \leq i \leq N$, the obtained box-scheme could be presented as the following:

$$-Q_i^{11}u_i^{(1)} + Q_i^{12}u_i^{(2)} + D_{i+1}^{11}u_{i+1}^{(1)} - D_{i+1}^{12}u_{i+1}^{(2)} = d_{i+1}^{(1)} \tag{7}$$

$$-Q_i^{21}u_i^{(1)} + Q_i^{22}u_i^{(2)} + D_{i+1}^{21}u_{i+1}^{(1)} - D_{i+1}^{22}u_{i+1}^{(2)} = d_{i+1}^{(2)} \tag{8}$$

Here the coefficients are

$$Q_i^{11} = S_- + r_-, \qquad Q_i^{12} = \alpha_- S_- + \gamma_- r_- \tag{9}$$

$$Q_i^{21} = \gamma_- S_- + \alpha_- r_-, \qquad Q_i^{22} = (S_- + r_-)\alpha_-\gamma_- \tag{10}$$

$$D_{i+1}^{11} = S_+ + r_+, \qquad D_{i+1}^{12} = \alpha_+ S_+ + \gamma_+ r_+ \tag{11}$$

$$D_{i+1}^{21} = \gamma_+ S_+ + \alpha_+ r_+, \qquad D_{i+1}^{22} = (S_+ + r_+)\alpha_+\gamma_+ \tag{12}$$

$$f_{i+1}^{(1)} = -\psi_-(1 - \alpha_- S_- - \gamma_- r_-) + \psi_+(1 - \alpha_+ S_+ - \gamma_+ r_+) \tag{13}$$

$$f_{i+1}^{(2)} = \alpha_-\gamma_-\psi_-(S_- + r_-) - \alpha_+\gamma_+\psi_+(S_+ + r_+) \tag{14}$$

The most general form of the boundary conditions is

$$D_1^{11}u_1^{(1)} - D_1^{12}u_1^{(2)} = d_1^{(1)} \tag{15}$$

$$-Q_{N+1}^{21}u_{N+1}^{(1)} + Q_{N+1}^{22}u_{N+1}^{(2)} = d_{N+2}^{(2)} \tag{16}$$

Regularization Algorithm

Here is the analysis of the regularization algorithm for the solution of the two-point equations (15), (7), (8), and (16). In this study an economical method of the regularization has been developed. This method uses significantly less arithmetic calculations than the identical method offered by Samarskii and Nikolaev.[4]

Consider two neighboring grid cells (i) and $(i + 1)$. In the cell (i) the boundary condition of the same type as Eq. (15) is

$$\bar{D}_i^{11}u_i^{(1)} - \bar{D}_i^{12}u_i^{(2)} = \bar{d}_i^{(1)} \tag{17}$$

Find the value $u_i^{(2)}$ from Eq. (8)

$$u_i^{(2)} = (Q_i^{22})^{-1}[G_{i+1}^{(2)} + Q_i^{21}u_i^{(1)}]$$
$$G_{i+1}^{(2)} = d_{i+1}^{(2)} - D_i^{21}u_{i+1}^{(1)} + D_{i+1}^{22}u_{i+1}^{(2)} \tag{18}$$

Substituting the parameter $u_i^{(2)}$ into Eq. (17), we find the value $u_i^{(1)}$:

$$u_i^{(1)} = \alpha_i^{(1)} + \gamma_i^{12}G_{i+1}^{2} \tag{19}$$

Thus, the value $u_i^{(2)}$ could be calculated as the following:

$$u_i^{(2)} = \alpha_i^{(2)} + \gamma_i^{22}G_{i+1}^{(2)} \tag{20}$$

In the case $1 \le i \le N$, the coefficients are

$$\alpha_i^{(1)} = [\bar{D}_i^{11} - \bar{D}_i^{12}(Q_i^{22})^{-1}Q_i^{21}]^{-1}\bar{d}_i^{(1)} \tag{21}$$

$$\alpha_i^{(2)} = (Q_i^{22})^{-1}Q_i^{21}\alpha_i^{(1)} \tag{22}$$

$$\gamma_i^{12} = [\bar{D}_i^{11} - \bar{D}_i^{12}(Q_i^{22})^{-1}Q_i^{21}]^{-1}\bar{D}_i^{12}(Q_i^{22})^{-1} \tag{23}$$

$$\gamma_i^{22} = (Q_i^{22})^{-1}(1 + Q_i^{21}\gamma_i^{12}) \tag{24}$$

In the grid cell $(N + 1)$ the system of equations will be obtained as the following:

$$\bar{D}_{N+1}^{11}u_{N+1}^{(1)} - \bar{D}_{N+1}^{12}u_{N+1}^{(2)} = \bar{d}_{N+1}^{(1)} \tag{25}$$

$$-Q_{N+1}^{21}u_{N+1}^{(1)} + Q_{N+1}^{22}u_{N+1}^{(2)} = d_{N+2}^{(2)} \tag{26}$$

The solution of Eqs. (25) and (26) is

$$u_{N+1}^{(1)} = \alpha_{N+1}^{(1)} + \gamma_{N+1}^{12}d_{N+1}^{(2)} \tag{27}$$

$$u_{N+1}^{(2)} = \alpha_{N+1}^{(2)} + \gamma_{N+1}^{22}d_{N+1}^{(2)} \tag{28}$$

The coefficients $\alpha_{N+1}^{(1)}$, γ_{N+1}^{12}, $\alpha_{N+1}^{(2)}$, and γ_{N+1}^{22} are determined by Eqs. (21–24) in the grid cell $i = N + 1$. The coefficients marked with ($\bar{\ }$) are calculated in the following way. If the bottom cell ($i = 1$) is at the boundary, then the coefficients $\bar{D}_1^{11}$, $\bar{D}_1^{12}$, and $\bar{d}_1^{(1)}$ are equal to the coefficients in Eq. (15). That is, $\bar{D}_1^{11} = D_1^{11}$, $\bar{D}_1^{12} = D_1^{12}$, and $\bar{d}_1^{(1)} = d_1^{(1)}$. The expressions for $u_i^{(1)}$ and $u_i^{(2)}$ from Eqs. (19) and (20) substitute the same parameters in Eqs. (7) and (8), when $i \ge 1$. Considering the structure of Eqs. (17) at $1 \le i \le N$, the new equations will be

$$\bar{D}_{i+1}^{11} = D_{i+1}^{11} + (Q_i^{11}\gamma_i^{12} - Q_i^{12}\gamma_i^{22})D_{i+1}^{21} \tag{29}$$

$$\bar{D}_{i+1}^{12} = D_{i+1}^{12} + (Q_i^{11}\gamma_i^{12} - Q_i^{12}\gamma_i^{22})D_{i+1}^{22} \tag{30}$$

$$\bar{d}_{i+1}^{(1)} = d_{i+1}^{(1)} + (Q_i^{11}\gamma_i^{12} - Q_i^{12}\gamma_i^{22})d_{i+1}^{(2)} + Q_i^{11}\alpha_i^{(1)} - Q_i^{12}\alpha_i^{(2)} \tag{31}$$

Here ends the direct stage of the regularization algorithm. The regularization coefficients will be used in the next (reverse) stage. Besides, the coefficients $\alpha_i^{(1)}$, γ_i^{12}, $\alpha_i^{(2)}$, γ_i^{22}, D_{i+1}^{21}, D_{i+1}^{22}, and $d_{i+1}^{(2)}$, obtained on the direct regularization stage, should be kept in mind.[6] Then, applying Eqs. (19) and (20), the reverse regularization stage can be performed.

The described method of regularization uses 37% less arithmetical calculations than the algorithm of Samarskii and Nikolaev.[4] The CPU timings also support this reduction by the factor of 1.32.

Uniform Convergence of Box-Schemes

The three-point box-scheme was offered by El-Mistikawy and Werle[3] for the solution of the boundary-value problems, which could be described by Eq. (5). They assumed that this box-scheme is uniform and of the second order. These properties of the box-scheme were proven by Doolan et al.[1] under the conditions $b = 0$. In this study, an accurate numerical analysis of the determination of the order of uniform convergence of both the two- and three-point box-schemes has been conducted by the method of Doolan et al.[1] Two steps of calculations were used.

Step A: The parameters $z_{k,\varepsilon}$ are calculated:

$$z_{k,\varepsilon} = \max|u_j^{h/2^k} - u_{2j}^{h/2^{k+1}}|, \quad k = 0, 1, \ldots \tag{32}$$

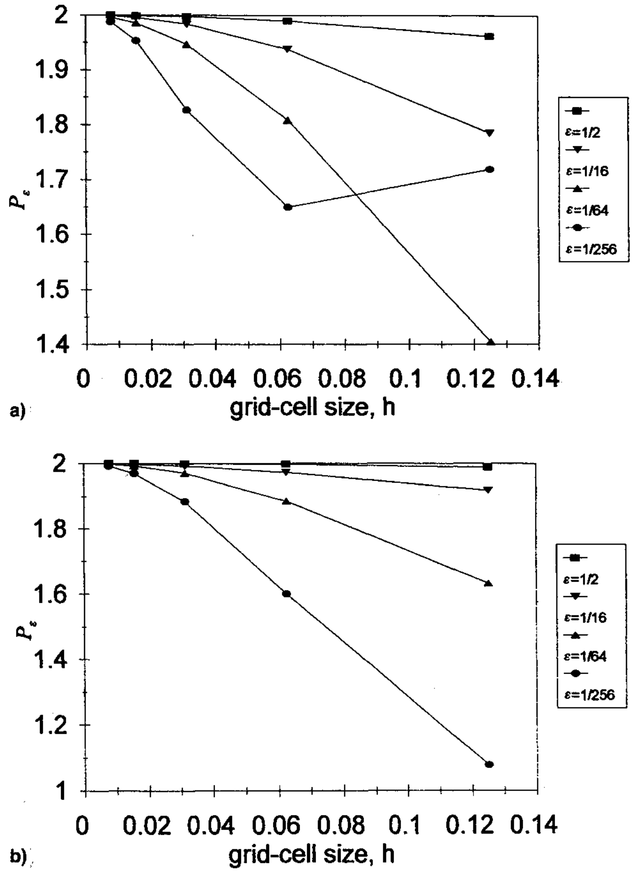

Fig. 1 Two-point box-scheme; P_ε vs h: a) $\bar{u}$ and b) $\bar{u}'$.

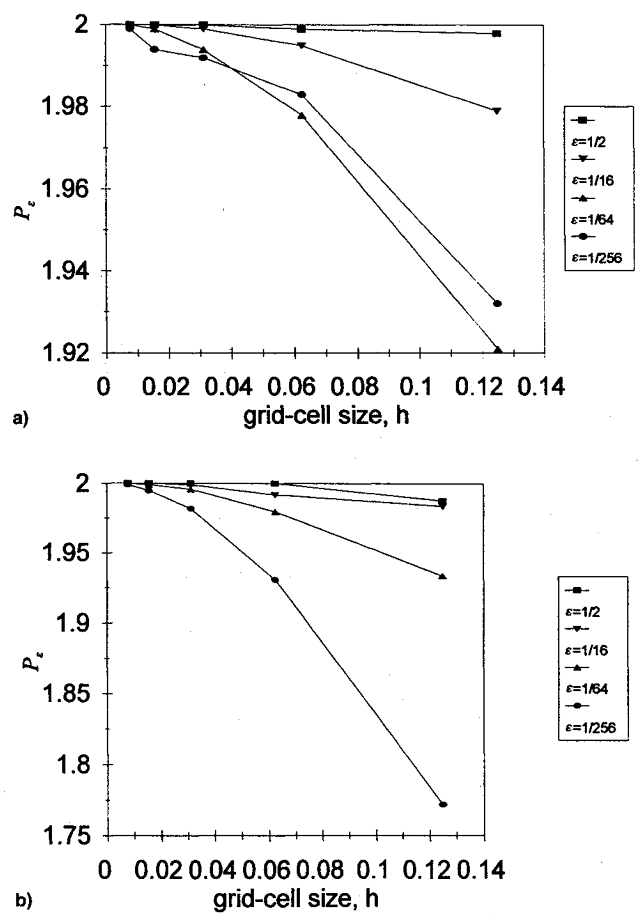

Fig. 2 Three-point box-scheme; P_ε vs h: a) $\bar{u}$ and b) $\bar{u}'$.

The maximum is estimated in all grid cells. The parameter h is the size of the largest cell.

Step B: The criteria of the uniformity is assessed:

$$z_{k,\varepsilon} \le C(h/2^k)^P, \qquad k = 0, 1, \ldots \tag{33}$$

Assume, that the criteria (33) is equivalent to the criteria

$$z_{k,\varepsilon} = C_\varepsilon(h/2^k)^{P_\varepsilon}, \qquad k = 0, 1, \ldots \tag{34}$$

where parameters C_ε and P_ε are independent of the parameter k, and therefore, the parameter P_ε could be estimated by formula

$$P_\varepsilon = \log_2(z_{k,\varepsilon}/z_{k+1,\varepsilon}) \tag{35}$$

Compared to the study of Doolan et al.[1] the calculations of the parameters in Eq. (35) have been conducted with double precision. The order of uniform convergence has been estimated by the considered procedure of calculations at constant ε and at $h \to 0$. The values of the parameter P_ε have been calculated for the linear boundary-value problem[1]:

$$-\varepsilon u''(x) + (1 + x^2)u(x) = (4x^2 - 14x + 4)(1 + x)^2 \tag{36}$$

$$u(0) - u'(0) = 0, \qquad u(1) + u'(1) = 0 \tag{37}$$

The magnitudes P_ε are shown in Figs. 1 and 2 for the two- and three-point box-schemes, respectively. The obtained data lead to the conclusion that both box-schemes are of the second order of the uniform convergence. From this point of view they are equivalent. The difference in the assessment of the order of uniformity is only in the coefficient of the terms remaining in the approximation. This should be expected due to the difference in the cell sizes and in the procedure for calculating the coefficients in Eq. (5), designing the box-scheme.

The two-point exponential box-scheme was used by Makashev and Provotorov[13] and Provotorov and Riabov[5,11] to solve the boundary-layer equations and the thin-viscous-shock-layer equations for nonequilibrium multicomponent dissociating gas-mixture flows. The results have been obtained in the whole range of chemical reaction rates up to the values near equilibrium. The iteration process permits one to obtain a rapid convergence towards the solution. This property of the numerical algorithm is especially significant when the influence of the recombination processes and blowing on the flow structure are essential. In the cases considered in previous studies[5,11,13] the number of iterations was below six.

Boundary-Layer Gas Flow with Blowing

The numerical method developed in this study has been used for the numerical solution of the boundary-layer equations under the conditions of moderate and intensive blowing from the body surface. The profiles (solid lines) of the tangential component of the velocity $\bar{u}$ and its derivative $\bar{u}'$ along the normal η at the stagnation point on the surface of the axisymmetrical blunt body ($\beta = 0.5$) are shown in Fig. 3 for different values of the blowing parameter ($f_w = 0, -2.5, -10$, and -25). As the rate of blowing increases, the boundary layer is thicker. The values of $\bar{u}'$ are decreased under these conditions. In this problem the surface of the body is considered as thermally isolated. Figure 3 also shows the results of the calculations obtained using the three-point exponential box-scheme (markers) developed by El-Mistikawy and Werle.[3] Comparison of the results obtained by different methods demonstrates that they are in a good agreement.

The flow parameters in the boundary layer near the cooling surface (at the temperature factor $t_w = 0.1$) with blowing are shown in Figs. 4 and 5 (solid lines). As discussed, the presence of the blowing flow significantly changes the structure of the flow in the boundary layer as seen in the distribution of the velocity $\bar{u}$. In addition, the essential changes in the distribution of the values $g = S/S_w$, which characterize the temperature changes, were obtained (see Fig. 5). For the $f_w = -10$ case, the main changes of the parameter g take place in the narrow zone $[1.25 < \log_{10}(\eta) < 1.8]$. The changes of the velocity take place in the significantly wider region $[0 < \log_{10}(\eta) < 1.8]$. As the temperature factor $t_w = 1 + S_w$ increases up to $t_w \approx 1$ (dashed lines in Figs. 4 and 5), the flow zone, formed by blowing, is located closer to the body surface. The transition boundary towards the upstream flow becomes wider. Without blowing, the influence of t_w on the distribution $\bar{u}$ and g is practically absent. The change of the blowing rate significantly influences the distribution of the velocity and temperature in the boundary layer, and as the result, this change significantly

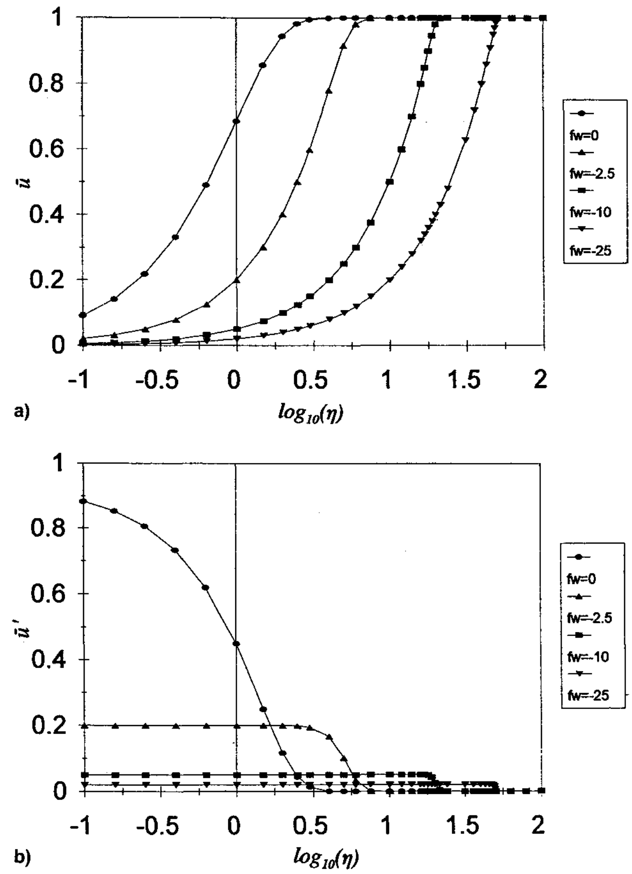

Fig. 3 Functions a) $\bar{u}$ and b) $\bar{u}'$ along η at the stagnation point for two-point box-scheme (lines). The markers show the three-point box-scheme data.

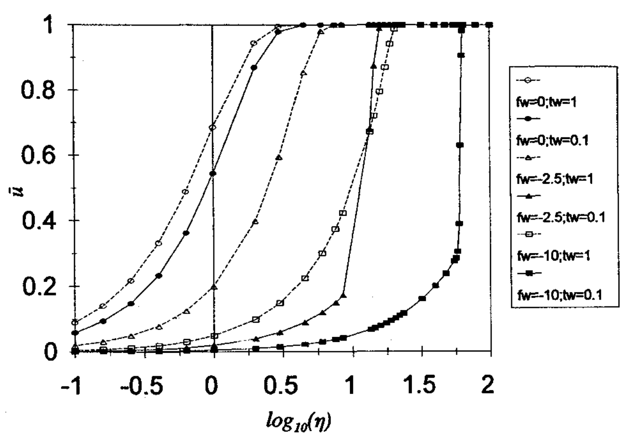

Fig. 4 Profile $\bar{u}$ along η at different values of $t_w = 0.1$ (solid lines) and $t_w = 1$ (dashed lines).

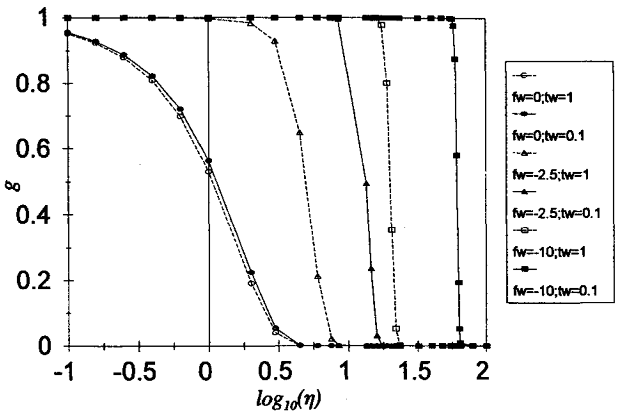

Fig. 5 Profile $g = S/S_w$ along η at different values of $t_w = 0.1$ (solid lines) and $t_w = 1$ (dashed lines).

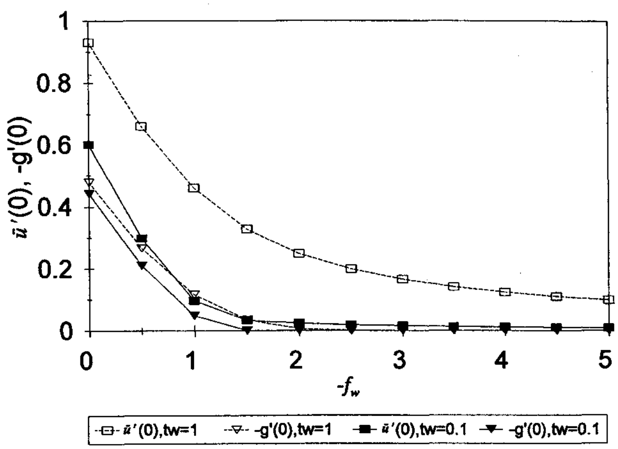

Fig. 6 Functions of friction $\bar{u}'(0)$ and heat flux $g'(0)$ vs mass flow rate of gas blowing f_w.

influences the heat flux and friction on the body surface. The complex structure of combustion zones in a thin viscous shock layer was discussed by Riabov and Botin[14] under the conditions of moderate Reynolds numbers and moderate hydrogen injections.

Figure 6 shows the parameters $g'(0)$ and $\bar{u}'(0)$, which characterize the heat flux and friction on the body surface as functions of the blowing parameter f_w. The body surface is turned to be fully thermally isolated at $|f_w| \geq 2$. Further increase of the parameter $|f_w|$ is not justified. The effectiveness of the thermal isolation by blowing increases at the decreasing of temperature factor t_w from 1.0 (dashed lines) to 0.1 (solid lines). At the same time the friction decreases.

Conclusions

The results of this study demonstrate that the usage of moderate blowing decreases the heat flux towards the body surface and also decreases the friction on the surface at the stagnation point. The flow parameters in this case should be calculated using the two-point exponential box-scheme. The major feature of this scheme is its uniform convergence of the second order, and it has to be considered in the full range of changing the blowing parameters. The application of the two-point box-scheme compared to the three-point box-scheme demonstrates essential advantages of the two-point exponential box-scheme.

References

[1]Doolan, E. P., Miller, J. J. H., and Schilders, W. H. A., *Uniform Numerical Methods for the Problems with Initial and Boundary Layers*, Boole Press, Dublin, 1980.

[2]Liu, T.-M., and Chiu, H. H., "Fast and Stable Numerical Method for Boundary-Layer Flow with Massive Blowing," *AIAA Journal*, Vol. 14, No. 1, 1976, pp. 114–116.

[3]El-Mistikawy, T. M., and Werle, M. J., "Numerical Method for Boundary Layers with Blowing—The Exponential Box Scheme," *AIAA Journal*, Vol. 16, No. 7, 1978, pp. 749–751.

[4]Samarskii, A. A., and Nikolaev, E. S., *Numerical Methods for Grid Equations*, Vols. 1 and 2, Birhäuser, Boston, 1989.

[5]Provotorov, V. P., and Riabov, V. V., "Using Exponential Difference Schemes for Describing Boundary Layers with Blowing," *Uchenyye Zapiski TsAGI*, Vol. 17, No. 1, 1986, pp.47–54 (in Russian).

[6]Provotorov, V. P., "A New Regularization Algorithm for Numerical Solution of the Thin-Viscous-Shock-Layer Equations," *Trudy TsAGI*, No. 2436, 1990, pp. 165–173 (in Russian).

[7]Schlichting, H., *Boundary-Layer Theory*, 7th ed., McGraw–Hill, New York, 1979.

[8]White, F. M., *Viscous Fluid Flow*, 2nd ed., McGraw–Hill, New York, 1991.

[9]Keller, H. B., "Accurate Difference Methods for Nonlinear Two-Point Boundary Value Problems," *SIAM Journal of Numerical Analysis*, Vol. 11, No. 2, 1974, pp. 305–320.

[10]Denisenko, O. V., and Provotorov, V. P., "Study of Viscous Gas Flows in Multicomponent Mixtures," *Trudy TsAGI*, No. 2269, 1985 (in Russian).

[11]Riabov, V. V., and Provotorov, V. P., "The Structure of Multicomponent Nonequilibrium Viscous Shock Layers," AIAA Paper 94-2054, June 1994.

[12]Provotorov, V. P., and Stepanov, E. A., "Numerical Analysis of Viscous Shock Layers with Blowing near the Stagnation Critical Point," *Uchenyye Zapiski TsAGI*, Vol. 16, No. 4, 1985, pp. 89–96 (in Russian).

[13]Makashev, N. K., and Provotorov, V. P., "Nonequilibrium Dissociation of Diatomic Molecules in the Boundary Layer," *Fluid Dynamics*, Vol. 20, No. 1, 1985, pp. 139–148.

[14]Riabov, V. V., and Botin, A. V., "Hypersonic Hydrogen Combustion in the Thin Viscous Shock Layer," *Journal of Thermophysics and Heat Transfer*, Vol. 9, No. 2, 1995, pp. 233–239.